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Mixed convection flow from a vertical flat plate with temperature dependent viscosity

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Abstract — A two-dimensional mixed convection flow of a viscous incompressible fluid of temperature dependent viscosity past a vertical impermeable fluid is considered. The governing equations for the flow are transformed for the regions appropriate to the forced convection, free convection and forced-free convection regimes. Solutions of the reduced equation appropriate in the forced convection and free convection regime are obtained using the perturbation technique treating ξ, the buoyancy parameter, as the perturbation parameter and those for the forced-free convection regime are obtained by the implicit finite difference method. Numerical results thus obtained are presented in terms of the local shear stress and local surface heat-flux. Effect of the viscosity variation parameter, ε, on the surface shear stress and the surface heat-flux for the fluid appropriate for Prandtl number ranging from 0.02 to 100 is shown. The perturbation solutions obtained for small and large values of ξ are found in excellent agreement with the finite difference solutions for the entire ξ regime. © 2000 Éditions scientifiques et médicales Elsevier SAS

free and forced convection / convective heat transfer / mixed convection / viscosity variation / perturbation

Nomenclature

x, y	coordinate measuring distance along the plate and normal to plate	m
u	velocity component in the x-direction	$\mathrm{m \cdot s^{-1}}$
v	velocity component in the y-direction	$\mathrm{m \cdot s^{-1}}$
f	dimensionless stream function	
g	gravitational acceleration	$\mathrm{m \cdot s^{-2}}$
T	temperature of the fluid	K
T_∞	temperature of the ambient fluid	K
T_w	plate temperature	K
C_p	specific heat at constant pressure	$\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$
U_∞	free stream velocity	$\mathrm{m \cdot s^{-1}}$
Pr_x	Prandtl number	
Gr_x	Grashof number	
Re_x	Reynolds number	

Greek symbols

α	thermal diffusivity	$\mathrm{m^2 \cdot s^{-1}}$
ψ	stream function	$\mathrm{s^{-1}}$
θ	dimensionless temperature function	
ξ	the local buoyancy parameter	
η	similarity variable	
τ_x	the local shear stress	Pa
q_x	the local surface heat flux	$\mathrm{W \cdot m^{-2}}$
κ	thermal heat conductivity	$\mathrm{W \cdot m^{-1} \cdot K^{-1}}$
ρ	fluid density	$\mathrm{kg \cdot m^{-3}}$
ν	the kinetic coefficient of viscosity	$\mathrm{m^2 \cdot s^{-1}}$
ε	viscosity variation parameter	
β	the coefficient of thermal expansion	$\mathrm{K^{-1}}$
μ	viscosity of the fluid	$\mathrm{kg \cdot m^{-1} \cdot s^{-1}}$
μ_∞	viscosity of the ambient fluid	$\mathrm{kg \cdot m^{-1} \cdot s^{-1}}$

1. INTRODUCTION

Free convection flow of viscous incompressible fluid past a vertical surface has been studied extensively because of its wide application in industry. Using momentum-integral methods, first, Pohlhausen [1] studied this problem. Similarity methods were used to formulate this problem by Ostrach [2] who applied numerical method to obtain solutions for various values of Prandtl number.

In a mixed convective flow the effects of both forced and free convection are found in comparable order. In many practical fields, we found significant temperature

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differences between the surface of the hot body and the free stream. These temperature difference cause density gradients in the fluid medium and in presence of gravitational force free convection affects become important.

The simplest physical model of such a flow is the two-dimensional laminar mixed convective flow along a vertical flat plate extensive studies of which had been conducted by Sparrow [3], Merkin [4], Lloyd and Sparrow [5], Wilks [6], Tingwei [7] and Raju et al. [8]. It has, generally, been recognized that ξ ($= Gr_x/Re_x^2$, where Gr_x is the Grashof number and Re_x the Reynolds number) is the governing parameter for the laminar boundary layer forced-free convective flow, which represents the ratio of buoyancy forces to the inertial forces inside the boundary layer. However, forced convection exists when the limit of ξ goes to zero, which occurs at the leading edge, and the free convection limit can be reached if ξ becomes large. Gebhart et al. [9] replaced the exponent of the Reynolds number by other values close to two in order to correlate there experimental results.

All the above studies were confined to the fluid with uniform viscosity. However, it is known that this physical property may change significantly with temperature. For instance, the viscosity of water decreases by about 240 percent when the temperature increases from 10 °C ($\mu = 0.0131\,\text{g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1}$) to 50 °C ($\mu = 0.00548\,\text{g}\cdot\text{cm}^{-1}\cdot\text{s}^{-1}$). To accurately predict the flow behavior, it is necessary to take into account this variation of viscosity. Recently, Gary et al. [10] and Mehta and Sood [11] have shown that when this effect is included, the flow characteristics may substantially be changed compared to constant viscosity. Recently, Kafoussius and Williams [12] and Kafoussius and Rees [13] have investigated the effect of temperature dependent viscosity on the mixed convection flow from a vertical flat plate in the region near the leading edge using the local nonsimilarity method. Very recently, Hossain and Kabir [14] investigated the natural convection flow from a vertical wavy surface with variable viscosity proportional to inverse linear function of temperature.

In the present study it is proposed to investigate the mixed convection flow of a viscous incompressible fluid having viscosity depending on temperature from an isothermal vertical flat plate the temperature of which is higher than that of the ambient fluid. In formulating the equations governing the flow under consideration, it has been assumed that the fluid property variations are limited to firstly the density, which is taken into account only in so far as its effects the buoyancy term (Boussinesq approximation) and secondly the viscosity. The viscosity of the fluid has been assumed to be inversely proportional to a linear function of temperature, a semi-empirical formula for the viscosity had been used by Ling and Dybbs [15]. The governing partial differential equations are reduced to locally nonsimilar partial differential equations by adopting appropriate transformations applicable to the forced convection, free convection and the entire forced-free convection regimes. Solutions of the equations for two extreme cases are obtained applying the perturbation method and those for the mixed convection regime by the use of the implicit finite difference method with Keller-box technique. Effects of the viscosity-variation parameter, ε, on the local shear-stress and the local surface heat flux are shown graphically as well as in tabular form for fluids of Prandtl number ranging from 0.02 to 100.0.

2. MATHEMATICAL FORMALISM

We consider the steady two-dimensional laminar free-forced convective flow of a viscous incompressible fluid along a semi-infinite vertical flat plate. The plate is located at the $x-z$ plane. The temperature of the plate T_w is uniform and higher than the free stream temperature T_∞. It is also assume uniform free stream velocity U_∞, parallel to the vertical plate. We further assume that property variation with temperature are limited to density and viscosity with the density taken into account only in so far as its effects the buoyancy term in the momentum equation (Boussinesq approximation) only. A schematic diagram illustrating the flow domain and the coordinate system is given in *figure 1*.

Under the above assumptions, the two dimensional boundary layer equations for the mixed convection flow of a fluid past the semi-infinite vertical plate are given below:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where u and v are the fluid velocity components along x- and y-axis which are parallel and normal to the plate, respectively, ρ is the density of the fluid, g the gravitational acceleration, β the coefficient of thermal expansion, T the temperature inside the boundary layer, and α is the thermal diffusivity.

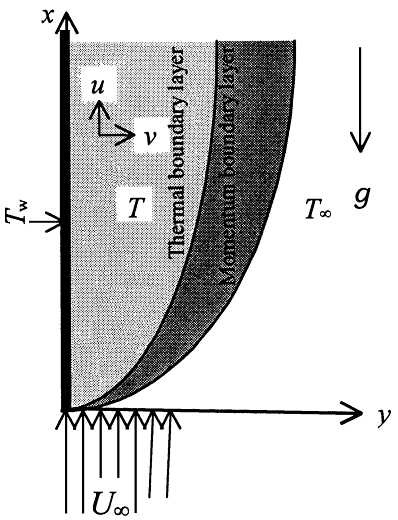

Figure 1. The flow configuration and the coordinates system.

The boundary conditions to be satisfied by the above equations are

$$\begin{aligned} &u=0, \quad v=0, \quad T=T_{\mathrm{w}} \quad \text{at } y=0 \\ &u=U_\infty, \quad T=T_\infty \quad \text{as } y\to\infty \end{aligned} \tag{4}$$

For the present fluid, following Ling and Dybbs [15], the viscosity is considered to be of the form

$$\mu=\frac{\mu_0}{1+\gamma(T-T_\infty)} \tag{5}$$

where μ_0 is the viscosity of the ambient fluid, γ is a constant.

For the fluid with constant viscosity (i.e., $\gamma=0$), the present problem has been studied by Merkin [4] and Hunt and Wilks [16]. Merkin, in his investigation, obtained appropriate transformations to reduce the equations (1)–(3) in the upstream region near the leading edge and in the downstream. Perturbation method has been employed to get the solutions in these two extreme regimes. He also obtained the solution of the transformed locally nonsimilar equations governing the flow in the entire forced-free convection regime using a step-by-step integration method developed by Choleshi [17].

3. TRANSFORMATIONS AND METHODS OF SOLUTION

Here we propose to transfer the equations (1)–(4) to convenient form valid in three different regimes, namely, (i) the forced convection regime, (ii) the free convection regime and (iii) the entire forced to free convection regime and discuss the solution methodologies of the transformed equations.

3.1. Forced convection regime

Since in the region near the leading edge, the flow is dominated by the forced convection, we may introduce the following group of transformations:

$$\begin{aligned} &\psi(x,y)=\nu_\infty Re_x^{1/2} f, && \frac{T-T_\infty}{T_{\mathrm{w}}-T_\infty}=\theta(\xi,\eta) \\ &\xi=\frac{Gr_x}{Re_x^2}, && \eta=\frac{y}{x}Re_x^{1/2} \\ &Gr_x=\frac{g\beta(T_{\mathrm{w}}-T_\infty)x^3}{\nu_\infty^2}, && Re_x=\frac{U_\infty x}{\nu_\infty} \end{aligned} \tag{6}$$

where ψ is the stream function that satisfies continuity equation (1) and is defined by

$$u=\frac{\partial\psi}{\partial y} \quad \text{and} \quad v=-\frac{\partial\psi}{\partial x} \tag{7}$$

ν_∞ $(=\mu_0/\rho)$ s the free stream kinetic viscosity, $f(\xi,\eta)$ is the dimensionless stream function, θ is the dimensionless temperature of the fluid in the boundary layer region, η is the pseudo-similarity variable and ξ is the local buoyancy parameter which measures the ratio of buoyancy forces to the inertial forces inside the boundary layer. The parameter ξ is small near leading edge where the forced convection dominates and large in the downstream where free convection dominates; Gr_x and Re_x are, respectively, the local Grashof number and the local Reynolds number.

Substitution of the transformations given in (6) into (1)–(4) one gets the following nonsimilar equations governing the flow and the energy distribution:

$$\begin{aligned} &(1+\varepsilon\theta)f'''-\varepsilon\theta' f''+(1+\varepsilon\theta)^2\left(\frac{1}{2}ff''+\xi\theta\right) \\ &\quad=\xi(1+\varepsilon\theta)^2\left\{f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}\right\} \end{aligned} \tag{8}$$

$$\frac{1}{Pr}\theta''+\frac{1}{2}\theta' f=\xi\left\{f'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial f}{\partial\xi}\right\} \tag{9}$$

where ε $(=(T_{\mathrm{w}}-T_\infty)\gamma)$ is termed as the viscosity-variation parameter which positive for heated surface and negative for cooled plate.

The boundary conditions appropriate for the above equation are

$$\begin{aligned} &f(\xi,0)=f'(\xi,0)=0, && \theta(\xi,0)=1 \\ &f'(\xi,\infty)=1, && \theta(\xi,\infty)=0 \end{aligned} \tag{10}$$

In equations (8) and (9), neglecting the terms that contain the derivatives with respect to ξ, the present problem reduces to that had been investigated by Hady et al. [18].

As mentioned above, the present problem for a fluid with constant viscosity (i.e., $\varepsilon = 0$) had been obtained by Merkin [4]. Since ξ is small in the forced convection regime, solutions of the equations (8) and (9) may be obtained by using the perturbation method treating this as perturbation parameter. Hence, we expand the functions $f(\xi,\eta)$ and $\theta(\xi,\eta)$ in powers of ξ as given below:

$$f(\xi,\eta) = \sum_{i=0}^{\infty} \xi^i f_i(\eta), \qquad \theta(\xi,\eta) = \sum_{i=0}^{\infty} \xi^i \theta_i(\eta) \quad (11)$$

Now, we substitute the above expansions into the equations (8)–(10) and take the terms upto O(ξ^2) to get following equations:

$$(1+\varepsilon\theta_0) f_0''' - \varepsilon\theta_0' f_0'' + \frac{1}{2}(1+\varepsilon\theta_0)^2 f_0 f_0'' = 0 \quad (12)$$

$$\frac{1}{Pr}\theta_0'' + \frac{1}{2}\theta_0' f_0 = 0 \quad (13)$$

$$\begin{aligned} &f_0(0) = f_0'(0) = 0, && \theta_0(0) = 1r \\ &f_0'(\infty) = 1, && \theta_0(\infty) = 0 \end{aligned} \quad (14)$$

$$\begin{aligned} &(1+\varepsilon\theta_0) f_1''' + \varepsilon\big(\theta_1 f_0''' - \theta_0' f_1'' - \theta_1' f_0''\big) \\ &\quad + \varepsilon(1+\varepsilon\theta_0)\theta_1 f_0 f_0'' \\ &\quad + (1+\varepsilon\theta_0)^2\left(\frac{1}{2} f_0 f_1'' + \frac{3}{2} f_1 f_0'' + \theta_0 - f_0' f_1'\right) = 0 \end{aligned} \quad (15)$$

$$\frac{1}{Pr}\theta_1'' + \frac{1}{2}\theta_1' f_0 + \frac{3}{2}\theta_0' f_1 - f_0'\theta_1 = 0 \quad (16)$$

$$\begin{aligned} &f_1(0) = f_1'(0) = \theta_1(0) = 0 \\ &f_1'(\infty) = \theta_1(\infty) = 0 \end{aligned} \quad (17)$$

$$\begin{aligned} &(1+\varepsilon\theta_0) f_2''' + \varepsilon\big(\theta_1 f_1''' + \theta_2 f_0''' - \theta_0' f_2' - \theta_1' f_1'' \\ &\quad - \theta_2' f_0''\big) + \frac{1}{2}\big(2\varepsilon(1+\varepsilon\theta_0)\theta_2 + \varepsilon^2\theta_1^2\big) f_0 f_0'' \\ &\quad + 2\varepsilon\theta_1(1+\varepsilon\theta_0)\left(\frac{1}{2} f_0 f_1'' + \frac{3}{2} f_1 f_0'' + \theta_0 - f_0' f_1'\right) \\ &\quad + (1+\varepsilon\theta_0)^2\left(\frac{1}{2} f_0 f_2'' + \frac{3}{2} f_1 f_1'' + \frac{5}{2} f_2 f_0'' \right. \\ &\quad \left. + \theta_1 - f_1'^2 - 2 f_0' f_2'\right) = 0 \end{aligned} \quad (18)$$

$$\frac{1}{Pr}\theta_2'' + \frac{1}{2}\theta_2' f_0 + \frac{3}{2}\theta_1' f_1 + \frac{5}{2}\theta_0' f_2 - 2 f_0'\theta_2 - f_1'\theta_1 = 0 \quad (19)$$

$$\begin{aligned} &f_2(0) = f_2'(0) = \theta_2(0) = 0 \\ &f_2'(\infty) = \theta_2(\infty) = 0 \end{aligned} \quad (20)$$

From the above, it can be seen that equations (14) and (15) are coupled and nonlinear by nature. Solutions of these equations are obtained by using the Natscheim–Swigert iteration technique together with sixth order implicit Runge–Kutta–Butcher initial value solver. The subsequent sets of equations are linear and hence solutions of these sets can easily be obtained. Here we have applied the method of superposition, which also known as the linear shooting method, in finding the solutions of the aforementioned sets for different values of the pertinent parameters.

Once the values of the functions f_n and θ_n for $n = 0, 1, 2$ and their derivatives at $\eta = 0$ are known, one can calculate the values of the local shearing stress and the local surface heat flux from the following relations:

$$\begin{aligned} \tau_x &= \left(\frac{\nu_\infty}{U_\infty g\beta\Delta T}\right)^{1/2} \frac{1}{\mu_0}\left(\mu\frac{\partial u}{\partial y}\right)_{y=0} \\ &= \frac{\xi^{-1/2} f''(\xi,0)}{1+\varepsilon} \end{aligned} \quad (21)$$

and

$$\begin{aligned} q_x &= \left(\frac{\nu_\infty U_\infty}{g\beta\Delta T}\right)^{1/2} \frac{1}{\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0} \\ &= \xi^{-1/2}\theta'(\xi,0) \end{aligned} \quad (22)$$

where $\Delta T = T_w - T_\infty$.

For example, taking $Pr = 10.0$ and $\varepsilon = 1.0$, the values of the local skin-friction and the local heat transfer rate may be obtained from the following expressions:

$$\begin{aligned} &(1+\varepsilon)\xi^{1/2}\tau_x \\ &\quad = 0.57613 + 1.05741\xi - 0.23692\xi^2 + \cdots \end{aligned} \quad (23)$$

and

$$\xi^{1/2} q_x = 0.83507 + 0.41553\xi - 0.23150\xi^2 + \cdots \quad (24)$$

The final values are entered in *tables I* and *II*, respectively, for comparison with other solutions.

3.2. Free convection regime

Far from the leading edge, that is in the free convection regime, the boundary layer is formed by buoyancy forces for which suggests transformations given by Ostrach [2] are as follows:

TABLE I
Numerical values of $f''(\xi,0)$ and $\theta'(\xi,0)$ taking $Pr = 0.7$ while $\varepsilon = 0.0$.

		$f''(\xi,0)$		$-\theta'(\xi,0)$	
$\xi_1 = (1-\xi)/\xi$	ξ	Raju et al. [8]	Present	Raju et al. [8]	Present
1.00000	0.00000	0.3321	0.3322	0.2928	0.2932
0.92237	0.29011	0.5919	0.5740	0.3373	0.3334
0.84935	0.42115	0.6889	0.6700	0.3505	0.3465
0.78075	0.52992	0.7536	0.7344	0.3584	0.3549
0.71639	0.62920	0.8020	0.7837	0.3639	0.3609
0.65610	0.72399	0.8404	0.8230	0.3680	0.3654
0.40960	1.20059	0.9743	0.9353	0.3823	0.3767
0.24010	1.77903	1.0171	0.9886	0.3828	0.3802
0.12960	2.59153	1.0282	1.0098	0.3794	0.3797
0.06250	3.87298	1.0251	1.0136	0.3751	0.3770
0.02560	6.16948	1.0145	1.0072	0.3706	0.3730
0.00810	11.06602	1.0002	0.9956	0.3662	0.3685
0.00160	24.97999	0.9848	0.9822	0.3619	0.3637
0.00010	99.99500	0.9699	0.9693	0.3576	0.3588
0.00000	∞	0.9570	0.9567	0.3531	0.3537

$$\begin{aligned} \psi &= \nu_\infty Gr_x^{1/4}\bar{f}(\xi,\bar{\eta}) \\ \bar{\eta} &= \frac{y}{x}Gr_x^{1/4} \\ \theta(\xi,\eta) &= \bar{\theta}(\xi,\bar{\eta}) \end{aligned} \tag{25}$$

Introducing the above transformations into the equations (2) and (3), and dropping the over-bars with brevity, one gets

$$(1+\varepsilon\theta)f''' - \varepsilon\theta' f'' + (1+\varepsilon\theta)^2\left(\frac{3}{4}ff'' - \frac{1}{2}f'^2 + \theta\right)$$
$$= \xi(1+\varepsilon\theta)^2\left\{f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right\} \tag{26}$$

$$\frac{1}{Pr}\theta'' + \frac{3}{4}\theta' f = \xi\left\{f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right\} \tag{27}$$

The boundary conditions (4) then become

$$\begin{aligned} &f(\xi,0) = f'(\xi,0) = 0, \qquad \theta(\xi,0) = 0 \\ &f'(\xi,\infty) = \xi^{-1/2}, \qquad \theta(\xi,\infty) = 0 \end{aligned} \tag{28}$$

The boundary condition (28) suggests the following series expansions for the functions f and θ involving the equations (26) and (27):

$$\begin{aligned} f(\xi,\eta) &= \sum_{i=0}^{\infty}\xi^{-i/2}f_i(\eta) \\ \theta(\xi,\eta) &= \sum_{i=0}^{\infty}\xi^{-i/2}\theta_i(\eta) \end{aligned} \tag{29}$$

TABLE II
Numerical values of $f''(\xi,0)$ and $\theta'(\xi,0)$ taking $Pr = 1.0$ while $\varepsilon = 0.0$.

	$f''(\xi,0)$		$\theta'(\xi,0)$	
ξ	Hunt and Wilks [16]	Present	Hunt and Wilks [16]	Present
0.0	0.46960	0.46954	0.46960	0.46954
0.2	0.66142	0.66353	0.50547	0.50861
0.4	0.78117	0.78288	0.52433	0.52748
0.6	0.86428	0.86544	0.53611	0.53893
0.8	0.92556	0.92630	0.54417	0.54666
1.0	0.97264	0.97307	0.55000	0.55221
2.0	1.10406	1.10470	0.56451	0.56979
4.0	1.19786	1.19720	0.57279	0.57681
5.0	1.21935	1.21848	0.57428	0.57769
7.0	1.24468	1.24370	0.57568	0.57823
10	1.26375	1.26280	0.57632	0.57809
15	1.27792	1.27722	0.57630	0.57775
20	1.28439	1.28358	0.57597	0.57706
30	1.28997	1.28929	0.57526	0.57595
60	1.29359	1.29318	0.57377	0.57407
10^2	1.29374	1.29349	0.57267	0.57282
10^3	1.28897	1.28900	0.56922	0.56921
10^4	1.28599	1.28613	0.56784	0.56781
10^6	1.28456	1.28454	0.56721	0.56717
10^{10}	1.28439	1.28435	0.56715	0.56711
10^{16}	1.28439	1.28435	0.56715	0.56711

Substituting the series (29) in equation (26) and (27) and taking the terms upto the O(ξ^1) we get the following sets of ordinary differential equations:

$$(1+\varepsilon\theta_0)f_0''' - \varepsilon\theta_0' f_0'' + (1+\varepsilon\theta_0)^2\left(\frac{3}{4}f_0 f_0'' - \frac{1}{2}f_0'^2 + \theta_0\right) = 0 \quad (30)$$

$$\frac{1}{Pr}\theta_0'' + \frac{3}{4}\theta_0' f_0 = 0 \quad (31)$$

$$\begin{aligned} &f_0(0) = f_0'(0) = \theta_0(0) = 0 \\ &f_0'(\infty) = \theta_0(\infty) = 0 \end{aligned} \quad (32)$$

$$\begin{aligned} &(1+\varepsilon\theta_0)f_1''' + \varepsilon\big(\theta_1 f_0''' - \theta_0' f_1'' - \theta_1' f_0''\big) \\ &+ 2\varepsilon(1+\varepsilon\theta_0)\theta_1\left(\frac{3}{4}f_0 f_0'' + \theta_0 - \frac{1}{2}f_0'^2\right) \\ &+ (1+\varepsilon\theta_0)^2\left(\frac{1}{4}f_1 f_0'' + \frac{3}{4}f_0 f_1'' + \theta_1 - \frac{1}{2}f_0' f_1'\right) = 0 \end{aligned} \quad (33)$$

$$\frac{1}{Pr}\theta_1'' + \frac{3}{4}\theta_1' f_0 + \frac{1}{4}\theta_0' f_1 + \frac{1}{2}f_0'\theta_1 = 0 \quad (34)$$

$$\begin{aligned} &f_1(0) = f_1'(0) = \theta_1(0) = 0 \\ &f_1'(\infty) = 1, \qquad \theta_1(\infty) = 0 \end{aligned} \quad (35)$$

$$\begin{aligned} &(1+\varepsilon\theta_0)f_2''' + \varepsilon\big(\theta_1 f_1''' + \theta_2 f_0''' - \theta_0' f_2'' \\ &- \theta_1' f_1'' - \theta_2' f_0''\big) + \varepsilon^2\theta_1^2\left(\frac{3}{4}f_0 f_0'' + \theta_0 - \frac{1}{2}f_0'^2\right) \\ &+ 2\varepsilon(1+\varepsilon\theta_0)\left(\frac{1}{4}f_1 f_1''\theta_1 + \frac{3}{4}f_0 f_1''\theta_1 + \theta_1^2 \right. \\ &\left. + \frac{3}{4}f_0 f_0''\theta_2 + \theta_0\theta_2 - \frac{1}{2}f'^2\theta_2\right) \\ &+ (1+\varepsilon\theta_0)^2\left(\frac{3}{4}f_0 f_2'' + \frac{1}{4}f_1 f_1'' - \frac{1}{4}f_2 f_0'' + \theta_2\right) = 0 \end{aligned} \quad (36)$$

$$\frac{1}{Pr}\theta_2'' + \frac{3}{4}\theta_2' f_0 + \frac{1}{4}\theta_1' f_1 - \frac{1}{4}\theta_0' f_2 + f_0'\theta_2 + \frac{1}{2}f_1'\theta_1 = 0 \quad (37)$$

$$\begin{aligned} &f_2(0) = f_2'(0) = \theta_2(0) = 0 \\ &f_2'(\infty) = \theta_2(\infty) = 0 \end{aligned} \quad (38)$$

The solutions of the above sets of equations are obtained using the method adopted in the preceding section.

As before, knowing the values of the functions f_n and θ_n (for $n = 0, 1, 2$) and their derivatives we can calculate the values of the local shearing stress and the local surface heat flux in the downstream regime from the following relations, which we get from the relations (21) and (22):

$$\tau_x = \frac{\xi^{1/4} f''(\xi, 0)}{1+\varepsilon} \quad \text{and} \quad q_x = \xi^{-1/4}\theta'(\xi, 0) \quad (39)$$

where the values of $f''(\xi, 0)$ and $\theta'(\xi, 0)$ may be obtained from the series representations given in (29).

For example, considering $Pr = 10.0$ and $\varepsilon = 1.0$, the values of the local skin-friction and the local heat transfer rate may be obtained from the following expressions:

$$(1+\varepsilon)\xi^{-1/4}\tau_x = 0.93993 - 0.10132\xi^{-1/2} + 169.533\xi^{-1} + \cdots \quad (40)$$

and

$$\xi^{1/4} q_x = 0.92292 + 0.0001\xi^{-1/2} + 0.00005\xi^{-1} + \cdots \quad (41)$$

In *tables I* and *II*, we have entered the asymptotic values of the local skin-friction and the local heat transfer are entered for comparison with other solutions.

Here also, substitutions for $\varepsilon = 0$ leads to the case investigated by Merkin [4] for the flow in the free convection regime. It should be noted that, instead of the present form of the series expansions of the functions, Merkin [4] considered a logarithmic expansion that was proposed by Stewartson [19]. But, it will be seen later that the present series solutions are in excellent agreement with finite difference solutions obtained for the downstream regime.

3.3. Forced and free convection regime

In order to obtain a system of equations applicable to the entire regime of mixed convection, we compare the transformations given in (6) and (23) and obtain

$$f = 2\xi^{1/4}\bar{f}, \qquad \eta = \bar{\eta}/\xi^{1/4}, \qquad \theta = \bar{\theta} \quad (42)$$

and, hence, a convenient switching from one system to the other at $\xi = 1$ facilitate successful integration over all ξ. Following Hunt and Wilks [16], we introduce the following continuous transformations to initiate the integration from forced to free convection regime:

$$\begin{aligned} &\psi(x, y) = \nu_\infty Re_x^{1/2}(1+\xi)^{1/4}\hat{f}(\xi, \hat{\eta}) \\ &\hat{\eta} = \frac{y}{x}Re_x^{1/2}(1+\xi)^{1/4} \end{aligned} \quad (43)$$

in equations (2)–(4). Dropping the hats with brevity, we anticipate the following system of equations:

$$(1+\varepsilon\theta) f''' - \varepsilon\theta' f'' + (1+\varepsilon\theta)^2 \left\{ \frac{2+3\xi}{4(1+\xi)} f f'' - \frac{\xi}{2(1+\xi)} f'^2 + \frac{\xi}{(1+\xi)}\theta \right\} = \xi(1+\varepsilon\theta)^2 \left\{ f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right\} \quad (44)$$

$$\frac{1}{Pr}\theta'' + \frac{2+3\xi}{4(1+\xi)}\theta' f = \xi \left\{ f' \frac{\partial \theta}{\partial \xi} - \theta' \frac{\partial f}{\partial \xi} \right\} \quad (45)$$

The boundary conditions appropriate for the above equation are

$$\begin{aligned} &f(\xi,0) = f'(\xi,0), && \theta(\xi,0) = 1 \\ &f'(\xi,\infty) = (1+\xi)^{-1/2}, && \theta(\xi,\infty) = 0 \end{aligned} \quad (46)$$

Clearly it can be seen that for small and large ξ the equations (44)–(46) reduces to the set of equations (8)–(10) and (26)–(28), respectively, those represent for forced convection regime and free convection dominated regime.

As before when the values of the functions f and θ and their derivatives are known, we can calculate the values of the local shearing stress and the local surface heat flux in the entire regime from the following relations, which we get using the definitions given in equations (21) and (22):

$$\tau_x = \frac{\xi^{-1/2}(1+\xi)^{3/4} f''(\xi,0)}{1+\varepsilon} \quad \text{and} \quad q_x = \xi^{-1/2}(1+\xi)^{1/4}\theta'(\xi,0) \quad (47)$$

Here the system of equations (44)–(46) have been simulated by using the implicit finite difference method which has, recently, been used most efficiently by Hossain et al. [14, 20]. According to this method, the system of partial differential equations (44) and (45) are first converted to a system of five first order partial differential equations by introducing new functions of η derivatives. This system is then put into finite-difference scheme in which the resulting nonlinear difference equations are linearized by the use of Newton's quasi-linearization method. The resulting linear difference equations along with the boundary conditions are finally solved by an efficient block-tridiagonal factorization method introduced by Keller [21]. Simulated results are obtained for a wide range of values of $\xi \in [0, 10^5]$ and discussed with effect of different pertinent parameters in the following section.

4. RESULTS AND DISCUSSIONS

Here we have investigated the problem of mixed convection flow of a viscous incompressible fluid with variable viscosity past a vertical impermeable and isothermal plate for three distinct flow regimes discussed above. Solutions are obtained for fluids having Prandtl number, $Pr = 0.02,\ 1.0,\ 10,\ 50$ and 100 against the local buoyancy parameter $\xi \in [0, 10^5]$ and for wide ranged values of the viscosity-variation parameter, $\varepsilon = 0.0, 1, 3, 6$ and 10, which are appropriate for heated surface.

The results are obtained in terms of the local skin-friction, τ_w, and the local surface heat flux, q_w, for different values of aforementioned physical parameters and these are shown in tabular form in *tables I* and *II*.

Numerical values of τ_x and q_x obtained for $\varepsilon = 0.0$ and $Pr = 0.7$ are entered into the *table I* for comparison with those of Raju et al. [8]. The comparison between these values are found in excellent agreement.

The values of $f''(\xi,0)$ and $\theta'(\xi,0)$ obtained for $Pr = 1.0$ at $\varepsilon = 0.0$ are entered into the *table II* for comparison with those of Hunt and Wilks [16]. To do this we have to multiply the values of $f''(\xi,0)$ and $\theta'(\xi,0)$ obtained from the present calculations with $\sqrt{2}$ because of the differences in transformation given in (43). In this comparison these values are found in excellent agreement.

The numerical values of the local shear stress τ_x and the local surface heat-flux q_x for various values of $\xi \in [0, 10^5]$ taking $Pr = 10,\ 50$ and 100 while $\varepsilon = 1$ obtained for three different flow regimes are depicted in *figures 2(a)* and *2(b)*, respectively. In these figures, the comparison between the series solutions for the forced convection regime (i.e. small ξ) and the free convection regime (i.e. for large ξ) shows in excellent agreement with the finite difference solutions obtained from the solutions of equations (44)–(46) for the entire forced to free convection regime. From these figures we further notice that in the entire forced to free convection regime the local shear-stress τ_x decreases and the local surface heat-flux q_x increases due to increase in the value of the Prandtl number Pr, which is expected.

Effect of viscosity-variation parameter ε on the local shear stress τ_x and the local surface heat flux q_x for a fluid having $Pr = 0.02$ are presented graphically in *figures 3(a)* and *3(b)*, respectively. In these figures, the dotted curves and the broken curves depict the results obtained by the series solutions of the equations for the forced convection regimes and the free convection regime, respectively. The comparison between these

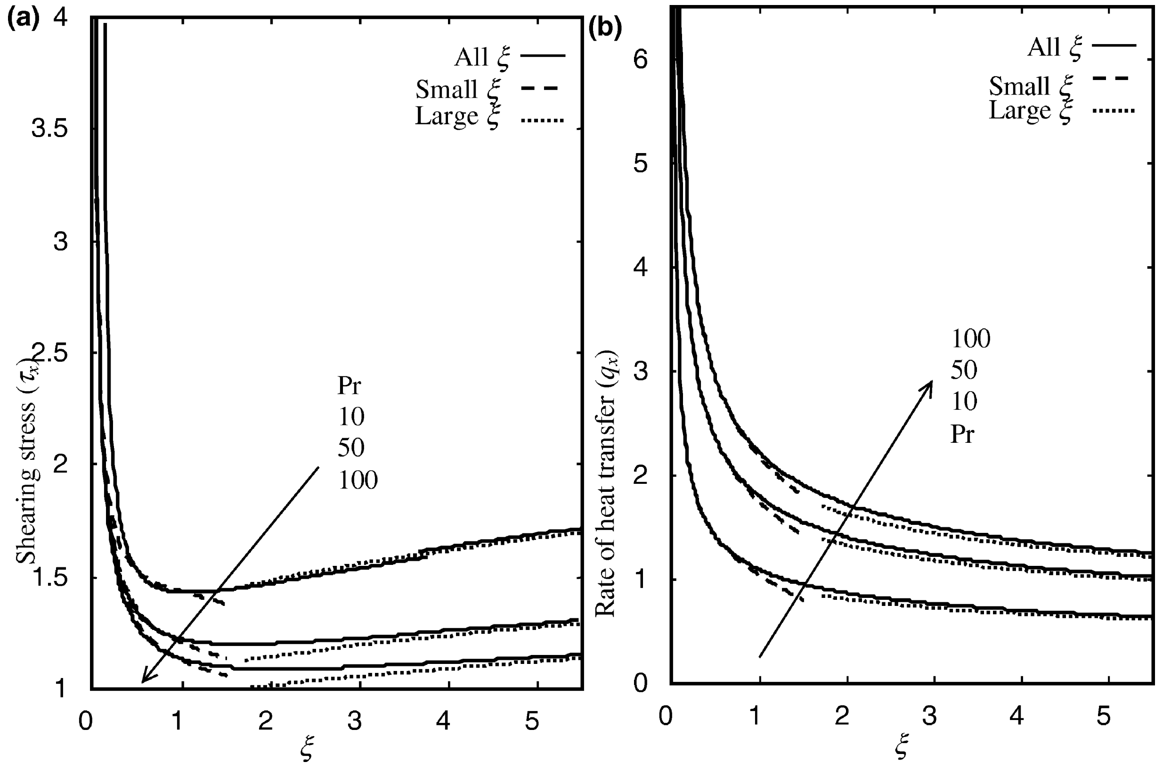

Figure 2. (a) The local shearing stress and (b) the rate of local heat flux while $Pr = 10.0,\ 50.0$ and 100.0 at $\varepsilon = 1.0$.

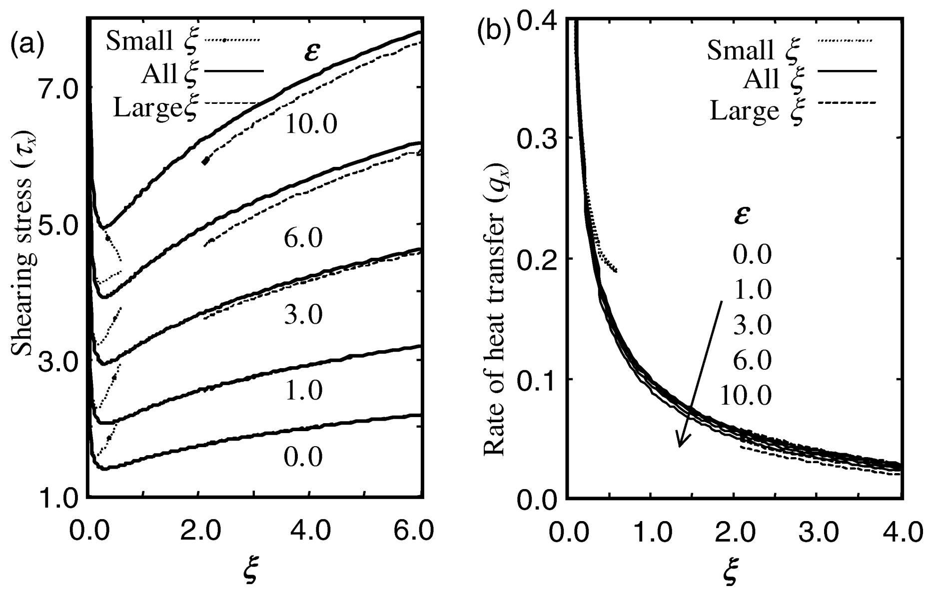

Figure 3. (a) The local shearing stress and (b) the rate of local heat flux for different $\varepsilon = 0.0,\ 1.0,\ 3.0,\ 6.0,\ 10.0$ while $Pr = 0.02$.

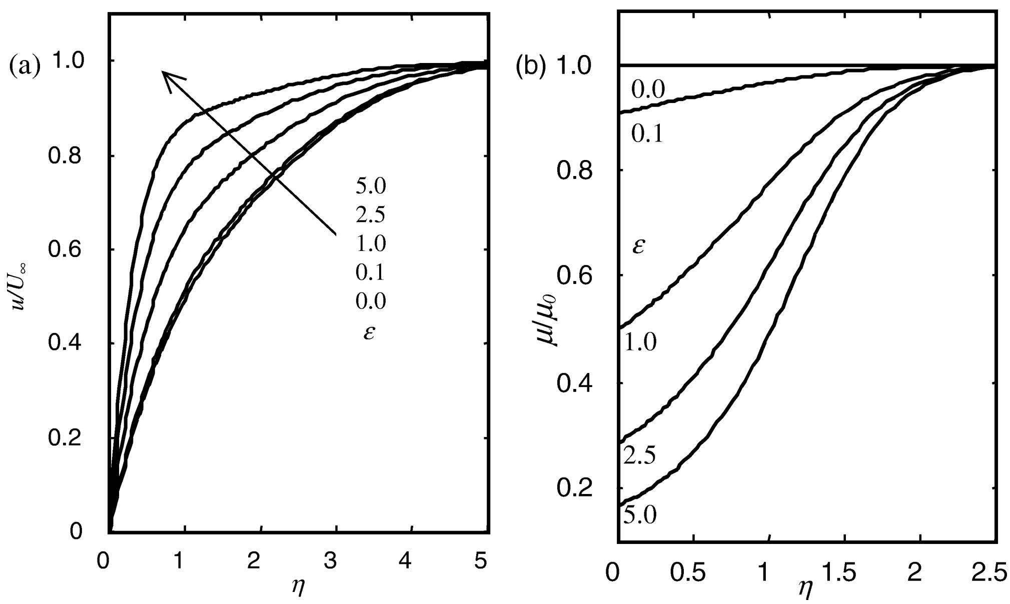

Figure 4. (a) The velocity profiles and (b) the viscosity profiles for different $\varepsilon = 5.0,\ 2.5,\ 1.0,\ 0.1,\ 0.0$ while $\xi = 1.0,\ Pr = 6.4$.

curves with the solids that represent the mixed convection solutions obtained by finite difference method are found in good agreement at each value of the viscosity-variation parameter ε. From these figures we may observe that an increase in the value of the viscosity-variation parameter leads to increase in the value of the local shear stress τ_x and to decrease in the value of the local rate of heat transfer q_x. It may further be seen that effect of the viscosity-variation parameter is very small on the increase in the rate of heat transfer q_x. From *figure 3(a)* we may observe that the value of skin-friction at the surface reaches to some minimum values at every value of ε near the plate. Such as for values of $\varepsilon = 0.0,\ 1.0,\ 3.0,\ 6.0$ and 10.0 the minimum values attained by the shear stress are found to at $\xi = 0.3045,\ 0.3045,\ 0.2837,\ 0.2837$ and 0.2837, respectively. From this we may conclude that these minimum values increase with the increase of the value of the viscosity-variation parameter ε.

Now we discuss the effect of the viscosity-variation parameter ε and the Prandtl number *Pr* on the velocity and the viscosity distributions obtained only from the finite difference solutions of the equations (44)–(46) that govern the mixed convection flow. The values of the velocity and the viscosity distributions have be calculated from the following relations:

$$\frac{u}{U_\infty} = (1+\xi)^{1/2} f(\xi, \hat{\eta}), \qquad \frac{\mu}{\mu_0} = \frac{1}{1+\varepsilon\theta} \tag{48}$$

Effect of increase in the viscosity variation parameter ε $(= 0.0,\ 1.0,\ 2.5,\ 5.0)$ on the nondimensional velocity u/U_∞ at $\xi = 1.0$ for the fluid with $Pr = 6.4$ which represents water at 20 °C and at 1 atm is shown in *figure 4(a)*. The corresponding profiles for the temperature-dependent viscosity, μ/μ_0, are depicted in *figure 4(b)*. From *figure 4(a)* it can be seen that increase in the value of the viscosity-variation parameter ε leads to increase in the velocity profile near the surface of the plate. Increase in the value of the viscosity-variation parameter ε also leads to increase in the viscosity near the surface of the plate and this approaches to unity value at the outer-edge of the boundary layer for every values of the viscosity-variation parameter considered here.

Figure 5(a) depicts the velocity profile for different values of the Prandtl number *Pr* $(= 1.0,\ 6.0,\ 10.0)$ while $\xi = 1.0$ and the viscosity variation parameter $\varepsilon = 5.0$. Corresponding distribution of the viscosity in the fluids considered are shown in *figure 5(b)*. From *figure 5(a)* it can be seen that increase in the value of the Prandtl number leads to decrease in the velocity profile near the surface of the plate. Finally, it may be observed from *figure 5(b)* that the viscosity of the fluid increases within the boundary layer region owing to increase in the value of the Prandtl number.

Solutions obtained from the present investigations could not be compared with any experimented data because of non-availability of any previous experimental

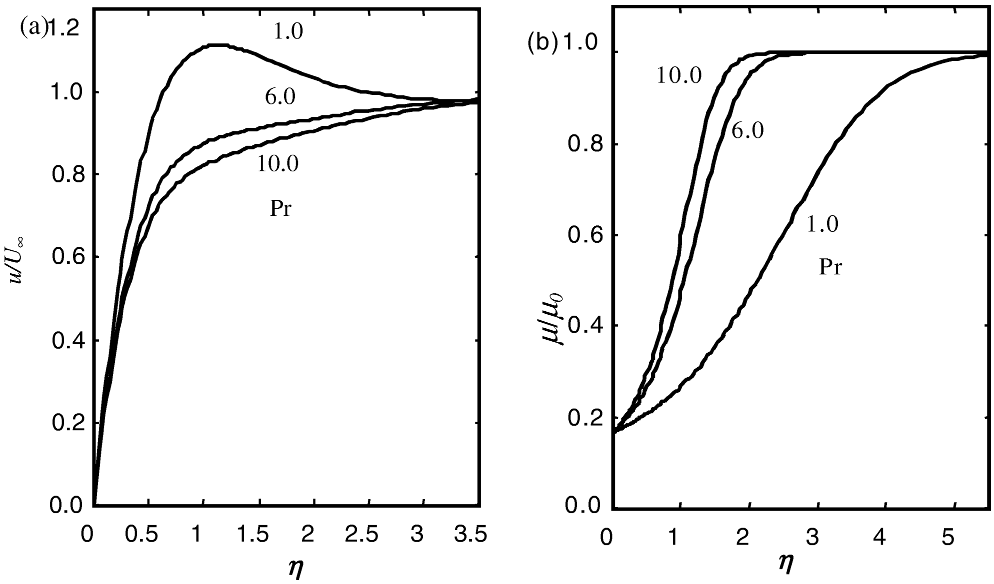

Figure 5. (a) The velocity profile and (b) the viscosity profile for different Prandtl number $Pr = 1.0,\ 6.0,\ 10.0$ while $\xi = 1.0$ and $\varepsilon = 5.0$.

result. But it is expected that the present results may be useful for future experiments.

5. CONCLUSIONS

The effects of temperature-dependent viscosity on the forced and free convection boundary layer flow along a vertical flat plate with uniform free stream and surface temperature has been investigated theoretically. The local nonsimilarity equations, governing the flow in the forced convection regime as well as in the free convection regime using the perturbation method. Numerical solutions to the equations governing the flow in the forced to free convection regime has also been obtained by the use of the implicit finite difference method. The comparison between the perturbation solutions obtained for the forced as well as free convection dominating regimes and the numerical solutions to the entire forced-free convection regime shows in excellent agreement. Solutions obtained from the present investigations could not be compared with any experimented data because of non-availability of any previous experimental result. But it is expected that the present results may be useful for future experiments. From the present investigation following conclusions may further be drawn:

1. In the mixed convection regime, the local shear stress τ_x increases and local rate of heat transfer q_x decreases as the value of buoyancy parameter ξ increases for all values of the Prandtl number Pr and the viscosity-variation parameter ε.

2. Increase in the value of the viscosity-variation parameter leads to increase in the local shear stress and to decrease in the local rate of heat transfer. Its effect on the increase of the rate of heat transfer is less than that of the local shear stress.

3. The velocity profiles increase and the viscosity of the fluid decrease near the surface of the plate owing to increase in the value of the viscosity-variation parameter.

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